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RELATIONS EXPRESSING HARMONIC DIVISION ON A STRAIGHT LINE.

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Chasles in his *Traite de Géométrie Supérieure*, Chap. IV, gives various relations expressing harmonic division on a straight line. Most of these relations are reproduced in this paper; but they have been deduced in each case in a manner different from that followed by Chasles.

1. General truths: If a, e, a', f are four points arranged in any order on a straight line, and if α, O , are the middle points of the segments aa', ef , and m any other point on the same line, then

$$ae + ea' + a'f + fa = 0, \quad (1)$$

$$af \cdot ea' + ae \cdot a'f + aa' \cdot fe = 0, \quad (2)$$

$$ma + ma' = 2ma, \quad (3)$$

$$ma \cdot ma' = \overline{ma}^2 - \overline{aa'}^2. \quad (4)$$

2. Definition of harmonic division: $\frac{ae}{af} \div \frac{a'e}{a'f} = -1$.

$$\therefore ae \cdot a'f = -a'e \cdot af. \quad (5)$$

By (5) and (2), $aa' \cdot ef = 2af \cdot ea', \quad (6)$

$$aa' \cdot fe = 2ae \cdot fa'. \quad (7)$$

These relations can also be deduced by the introduction of the point at infinity, i . For $\frac{ia}{ia'} \div \frac{aa}{aa'} = 2$, and $\frac{ia}{ia} \div \frac{a'a}{a'a} = 2$. These relations involve only anharmonic ratios; they are therefore true when for the harmonically conjugate points, i, α , we substitute the points e, f .

$$\therefore \frac{ef}{ea'} \div \frac{af}{aa'} = 2, \quad \frac{ef}{ea} \div \frac{a'f}{a'a} = 2,$$

or

$$aa' . ef = 2af . ea', \quad (6)$$

$$aa' . fe = 2ae . fa'. \quad (7)$$

$$3. \quad -1 = \frac{\frac{af}{af}}{\frac{a'e}{ae}} = \frac{\frac{af}{aa' . af}}{\frac{a'e}{aa' . ae}} = \frac{\frac{af - aa'}{aa' . af}}{\frac{ae - aa'}{aa' . ae}} = \frac{\frac{I}{aa'} - \frac{I}{af}}{\frac{I}{aa'} - \frac{I}{ae}};$$

$$\therefore \frac{2}{aa'} = \frac{I}{ae} + \frac{I}{af}. \quad (8)$$

$$4. \quad mf . a'e + ma' . ef + me . fa' = 0;$$

$$\begin{aligned} \therefore 2ma' &= me . \frac{2fa'}{fe} + mf . \frac{2ea'}{ef} \\ &= me . \frac{aa'}{ae} + mf . \frac{aa'}{af}, \text{ by (6), (7);} \\ \therefore \frac{2ma'}{aa'} &= \frac{me}{ae} + \frac{mf}{af}. \end{aligned} \quad (9)$$

5. Multiply (6), (7):—

$$-\overline{fe}^2 . \overline{aa'}^2 = 4fa . fa' . ea . ea';$$

$$\begin{aligned} \therefore -\overline{fe}^2 . \overline{aa}^2 &= (\overline{fa}^2 - \overline{aa}^2)(\overline{ae}^2 - \overline{aa}^2), \text{ by (4),} \\ &= \overline{fa}^2 . \overline{ea}^2 - \overline{aa}^2(\overline{fa}^2 + \overline{ea}^2) + \overline{aa}^4. \end{aligned}$$

But

$$\begin{aligned} \overline{fe}^2 &= (ae - af)^2 = \overline{ae}^2 + \overline{af}^2 - 2ae . af; \\ \therefore 0 &= \overline{af}^2 . \overline{ae}^2 - 2ae . af . \overline{aa}^2 + \overline{aa}^4; \\ \therefore \overline{aa}^2 &= af . ae. \end{aligned} \quad (10)$$

Similarly,

$$\overline{Oe}^2 = Oa . Oa'. \quad (11)$$

This relation may also be deduced thus:—

$$\frac{ea}{ea'} = -\frac{fa}{fa'}.$$

$$\therefore \frac{Oa - Oe}{Oa' - Oe} = -\frac{Oa - Of}{Oa' - Of} = -\frac{Oa + Oe}{Oa' + Oe}.$$

$$\therefore \overline{Oe}^2 = Oa . Oa'.$$

$$\begin{aligned} 6. \quad ea . ea' &= \overline{ea}^2 - \overline{aa}^2 = \overline{ea}^2 - af . ae, \text{ by (10),} \\ &= ea(ea + af) = ea . ef; \end{aligned}$$

$$\text{i. e.} \quad ea \cdot ea' = eu \cdot ef. \quad (12)$$

$$\begin{aligned} 7. \text{ By (11),} \quad \bar{Oe}^2 &= Oa \cdot Oa' = \bar{Oa}^2 - \bar{aa}^2, \text{ by (14);} \\ \therefore \bar{Oa}^2 - \bar{aa}^2 - \bar{Oe}^2 &= 0; \\ \therefore (ma - mO^2) - \bar{aa}^2 - \bar{Oe}^2 &= 0; \\ \therefore (\bar{ma}^2 - \bar{aa}^2) + (\bar{mO}^2 - \bar{Oe}^2) &= 2ma \cdot mO; \\ \therefore ma \cdot ma' + me \cdot mf &= 2ma \cdot mO. \end{aligned} \quad (13)$$

$$\begin{aligned} 8. \text{ By (12),} \quad ma \cdot ma' &= ma \cdot mm', \\ me \cdot mf &= mO \cdot mm'', \\ m, a, m', a'; m, e, m'', f &\text{ being harmonic ranges;} \end{aligned}$$

$$\begin{aligned} \therefore \frac{1}{ma \cdot ma'} + \frac{1}{me \cdot mf} &= \frac{1}{mm' \cdot mm''} \left(\frac{mm'' \cdot mO + mm' \cdot ma}{ma \cdot mO} \right) \\ &= \frac{1}{mm' \cdot mm''} \left(\frac{me \cdot mf + ma \cdot ma'}{ma \cdot mO} \right) \\ &= \frac{2}{mm' \cdot mm''}, \text{ by (13);} \end{aligned}$$

$$\text{i. e.,} \quad \frac{1}{ma \cdot ma'} + \frac{1}{me \cdot mf} = \frac{2}{mm' \cdot mm''}. \quad (14)$$

9. To show, without introducing the point at infinity, that

$$\frac{ma \cdot ma'}{na \cdot na'} + \frac{me \cdot mf}{ne \cdot nf} = 2 \frac{nn' \cdot nn''}{nn' \cdot nn''},$$

n being any point, and n, a, n', a' ; n, e, n'', f , harmonic ranges.

$$\frac{1}{nn'} + \frac{1}{nn''} = \frac{nn' + nn''}{nn' \cdot nn''};$$

$$\text{i. e.} \quad \frac{na + na'}{na \cdot na'} + \frac{ne + nf}{ne \cdot nf} = 2 \frac{nn' + nn''}{nn' \cdot nn''}, \text{ by (8);}$$

$$\begin{aligned} \therefore \frac{mn(na + na') + na \cdot na'}{na \cdot na'} + \frac{mn(ne + nf) + ne \cdot nf}{ne \cdot nf} \\ = 2 \frac{mn(nn' + nn'') + nn' \cdot nn''}{nn' \cdot nn''}. \end{aligned}$$

$$\text{Also, by (14),} \quad \frac{\overline{mn}^2}{na \cdot na'} + \frac{\overline{mn}^2}{ne \cdot nf} = 2 \frac{\overline{mn}^2}{nn' \cdot nn''};$$

$$\therefore \frac{\overline{mn}^2 + mn(na + na') + na \cdot na'}{na \cdot na'} + \frac{\overline{mn}^2 + mn(ne + nf) + ne \cdot nf}{ne \cdot nf}$$

$$= 2 \frac{\overline{mn}^2 + mn(nn' + nn'') + nn' \cdot nn''}{nn' \cdot nn''};$$

$$\text{i. e. } \frac{(mn + na)(mn + na')}{na \cdot na'} + \frac{(mn + ne)(mn + nf)}{ne \cdot nf} = 2 \frac{(mn + nn')(mn + nn'')}{nn' \cdot nn''};$$

$$\text{i. e. } \frac{ma \cdot ma'}{na \cdot na'} + \frac{me \cdot mf}{ne \cdot nf} = 2 \frac{mn' \cdot mn''}{nn' \cdot nn''}. \quad (15)$$

$$\begin{aligned} 10. \quad ma \cdot ma' + me \cdot mf &= 2 ma \cdot mO \\ &= 2 (me + ea) mO \\ &= 2 me \cdot mO + 2ea \cdot mO \\ &= me (me + mf) + 2ea \cdot mO \\ &= \overline{me}^2 + me \cdot mf + 2ea \cdot mO \\ \therefore ma \cdot ma' - \overline{me}^2 + 2ae \cdot mO &= 0. \end{aligned} \quad (16)$$

11. Multiply the preceding equation by ef :

$$\begin{aligned} ma \cdot ma' \cdot ef + \overline{me}^2 \cdot fe - 2ae \cdot mO \cdot fe &= 0; \\ \therefore ma \cdot ma' \cdot ef + \overline{me}^2 (fa + ae) - ae \cdot (me + mf)(me - mf) &= 0; \\ \therefore ma \cdot ma' \cdot ef + \overline{me}^2 \cdot fa + \overline{mf}^2 \cdot ae &= 0. \end{aligned} \quad (17)$$

12. To deduce the relation

$$\frac{ef \cdot nn'}{na \cdot na'} + \frac{fn'}{ne} + \frac{n'e}{nf} = 0,$$

n, a, n', a' ; a, e, a', f being harmonic ranges, without introducing the point at infinity.

$$\text{By (14), } \frac{1}{na \cdot na'} + \frac{1}{ne \cdot nf} = \frac{2}{nn' \cdot nn''} = \frac{2nO}{nn' \cdot nn'' \cdot nO} = \frac{ne + nf}{nn' \cdot ne \cdot nf};$$

$$\therefore \frac{nn'}{na \cdot na'} + \frac{nn'}{ne \cdot nf} - \frac{ne + nf}{ne \cdot nf} = 0;$$

$$\therefore \frac{nn' \cdot ef}{na \cdot na'} + nn' \left(\frac{nf - ne}{ne \cdot nf} \right) - \frac{\overline{nf}^2 - \overline{ne}^2}{ne \cdot nf} = 0;$$

$$\therefore \frac{nn' \cdot ef}{na \cdot na'} + nn' \left(\frac{1}{ne} - \frac{1}{nf} \right) - \frac{nf}{ne} + \frac{ne}{nf} = 0;$$

$$\begin{aligned} \therefore \frac{nn' \cdot ef}{na \cdot na'} + \frac{1}{nc} (nn' - nf) + \frac{1}{nf} (nc - nn') &= 0; \\ \therefore \frac{nn' \cdot ef}{na \cdot na} + \frac{fn'}{nc} + \frac{n'e}{nf} &= 0. \end{aligned} \quad (18)$$

13. m being any point, and $a, e, a', f; n, a, n', a'$ harmonic ranges, to show without introducing the point at infinity, that

$$\begin{aligned} \frac{ma \cdot ma'}{na \cdot na'} \cdot ef \cdot nn' + \frac{\overline{me}^2}{\overline{ne}^2} \cdot fn' \cdot nc + \frac{\overline{mf}^2}{\overline{nf}^2} \cdot n'e \cdot nf &= 0. \\ ef \cdot nn' + nc \cdot fn' + n'e \cdot nf &= 0, \text{ by (2),} \\ \overline{mn}^2 \left(\frac{ef \cdot nn'}{na \cdot na} + \frac{fn'}{nc} + \frac{n'e}{nf} \right) &= 0, \text{ by (18),} \\ \frac{nn' \cdot (na + na')}{na \cdot na'} \cdot ef &= 2ef = -2(fn' + n'e), \text{ by (12), (3),} \\ \text{or } mn \left(\frac{nn' (na + na')}{na \cdot na'} \cdot ef + 2fn' + 2n'e \right) &= 0. \end{aligned}$$

Adding these three equations, and arranging the terms,

$$\begin{aligned} \frac{\overline{mn}^2 + mn(na + na') + na \cdot na'}{na \cdot na'} \cdot ef \cdot nn' + \frac{\overline{mn}^2 + 2mn \cdot nc + \overline{ne}^2}{nc} \cdot fn' \\ + \frac{\overline{mn}^2 + 2mn \cdot nf + \overline{nf}^2}{nf} \cdot n'e &= 0. \\ \text{i. e., } \frac{(mn + na)(mn + na')}{na \cdot na'} \cdot ef \cdot nn' + \frac{(mn + nc)^2}{\overline{nc}^2} \cdot fn' \cdot nc \\ + \frac{(mn + nf)^2}{\overline{nf}^2} \cdot n'e \cdot nf &= 0; \end{aligned}$$

$$\text{i. e., } \frac{ma \cdot ma'}{na \cdot na'} \cdot ef \cdot nn' + \frac{\overline{me}^2}{\overline{ne}^2} \cdot fn' \cdot nc + \frac{\overline{mf}^2}{\overline{nf}^2} \cdot n'e \cdot nf = 0. \quad (19)$$